RAN-1901001101040001

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M.A. (Part -I) Examination March - 2026

Mathematics (Paper - 404) Ordinary Differential Equations

Time: 3 Hours] **[Total Marks: 100**

સૂચના : / Instructions

(1) ઉપરોક્ત દર્શાવેલ વિગતો ઉત્તરવહી પર અવશ્ય લખવી.
Fill up strictly the above details on your answer book
(2) Attempt ALL questions.
(3) Figures to the right indicate marks of the questions.
(4) Follow usual notations.

Seat No.:

Student's Signature

Q.1 (I) A system $x'(t) = A(t)x(t)$ is restrictively stable iff it is reducible to zero. **(07)**

(II) If $y' = -A^T(t)y\, f(t, x)$ is stable and $\int_0^t Tr(s)\, ds \le d < \infty$, *for* $t \ge 0$, then prove that it is restrictively stable. **(07)**

(III) If $u = \phi(t)$ is a solution of the Riccati equation $u' = p(t)u + q(t)u^2 + r(t)$, show that this equation has other solution of the form $u = \phi(t) - \dfrac{1}{\psi(t)}$ where $\psi(t)$ is a solution of an equation of the form $u' = a(t)u + b(t)$. **(06)**

OR

Q.1 (I) Let there exists a positive constant M such that $\|\Phi(t)\Phi^{-1}(s)\| \le M$, for $t_0 \le s < t < \infty$ and let $f(t, x)$ satisfies the inequality $\|f(t, x)\| \le \gamma(t)\|x\|$ where $\gamma(t)$ is non-negative continuous function such that $\int_{t_0}^{\infty} \gamma(t)\, dt < \infty$. Then there exists a positive constant L such that if $t \ge t_0$ every solution $x(t)$ of $x' = A(t)x + f(t, x)$ for which $\|x(t_1)\| \le C/M$ is defined $\forall t_1 \ge t_0$ and satisfies $\|x(t)\| \le M_1\, e^{-\alpha(t-t_1)}\|x(t_1)\|$, $\forall t \ge t_1$ and for some $M_1 > 0$. **(07)**

(II) If $x'(t) = A(t)\, x(t)$ is stable and $\int_0^t T_r A(s) ds \ge -d > -\infty$ for $t_0 \ge 0$ then it is restrictively stable **(07)**

(III) For the differential equation u' = u(l–u) show that (a) the solution u(t) = 0 is unstable (b) the solution u(t) = 1 is asymptotically stable. **(06)**

Q.2 (I) If $\|u\|$ *and* $\|u''\|$ are bounded then $\|u'\|$ is also bounded. **(07)**

(II) Show that the critical point (0,0) of the Vander-Pol's equation $u'' + \varepsilon(u^2 - 1)u' + u = 0$, where ε is a positive constant, is always unstable **(07)**

(III) Show that $u(t) = \exp\left(\int_1^t \frac{\sin^2 s}{s} ds\right) Sin\, t$ is a solution of $u'' + (1 + b(t))u = 0$ where $b(t) = 2\, \frac{sin2t}{t}\left(\frac{sin2t}{8t} - 1\right)$, $t > 0$ but this solution is not bounded **(06)**

OR

Q.2 (I) Let $b(t)$ be the continuously differentiable on $[0,\infty)$. If $b(t) \to 0$ as $t \to \infty$ and $\int_0^{\infty} |b'(s)|\, ds < \infty$, then all the solutions of $u'' + (1 + b(t))u = 0$ are bounded over $[0, \infty)$. **(07)**

(II) If all the solutions of $u'' + a(t)u = 0$ belongs to $L^2\,[0, \infty)$ and b(t) is bounded on $[0,\infty)$ then the solutions of $u'' + (a\,(t) + b(t))u = 0$ also belongs to $L^2[0,\infty)$. **(07)**

(III) Define stable characteristic polynomial. Determine, the stable characteristic polynomial for the following equation. **(06)**

(i) $y''' + 2y'' + 3y' + 4y = 0$

Q.3 (I) If g(t, u) is a continuous function of t and u in a Closed bounded region R(a, b) and satisfies the Lipschitz condition in R then there exists a unique solution u(t) to the initial value problem $u' = g(t,u)$, $u(t_0) = u_0$ on the interval J: $|t-t_0| \le h$, where $h = \min(a, \frac{b}{M})$ and $|g(t,u(t)| \le M$ for $(t,u) \in D$ and D: $|t-t_0| \le h$; $|u-u_0| \le b$ **(07)**

(II) If g(t, u) is defined in a rectangle $R(a, b) : t_0 \le t \le t_0 + a$, $|u - u_0| \le b$ and also g is measurable in t. If there exists a Lebesgue integrable function m(t) on $[t_0, t_0+a]$ such that $|g\,(t,u(t)| \le m(t)$ for $(t,u) \in R(a,b)$ then the initial value problem $u' = g(t,u), u(t_0) = u_0$ has at least one solution u(t) on some interval $[t_0, t_0 + \beta]$, $\beta > 0$. **(07)**

(III) Let u and v be non-negative continuous functions on $t_0 \le t \le t_0 + a$ satisfying $u(t) \le c + \int_{t_0}^{t} u(s)v(s)ds$, $t \in [t_0, t_0 + a]$, where c is a non-negative constant, then the inequality $u(t) \le c \exp\left(\int_{t_0}^{t} v(s)ds\right)$, $t \in [t_0, t_0 + a]$ holds. **(06)**

OR

Q.3 (I) Let a(t) and b(t) be continuous on the interval I and let $t_0 \in I$ then there exists a unique solution u(t) to the initial value problem $u' = a(t)u + b(t)$; $u(t_0) = u_0$ on I. **(07)**

(II) If g(t, u) is a continuous function in a strip $S : t_0 \le t \le t_0 + a$, $|u| < \infty$ and g is monotonically non-increasing in u for each fixed t in S then the initial value problem $u' = g(t,u)$, $u(t_0) = u_0$ has a unique solution u(t) defined on the interval $[t_0, t_0 + a]$. **(07)**

(III) Show that the following function satisfy the Lipschitz's condition. **(06)**

$g(t, u) = 10t^2\, e^{-u^3}$; R: $|t| \le 2$, $|u| \le 1$

Q.4 (I) The fundamental matrix $\Phi(t)$ is the solution of the matrix differential equation $\Phi'(t) = A(t)\Phi\,(t)$ and satisfies $\Phi(t_0) = I$. Further the solution x(t) of $x' = A(t)x$ satisfying $x(t_0) = x_0$ can be written as $x(t) = \Phi(t_0)x_0$. **(07)**

(II) Define fundamental matrix of $x'(t) = A(t)x(t)$. Prove that if $\Phi(t)$ is a fundamental matrix of $x'(t) = A(t)x(t)$ and C is any constant nonsingular matrix, then $\Phi(t)C$ is also a fundamental matrix of $x'(t) = A(t)x(t)$. **(07)**

(III) If $0 < 2\beta < \alpha < 1$ prove that there is a unique continuously differentiable positive function u(t) on $0 \le t \le 1$ such that $u' = \alpha u + \beta u(1) + 1$, $u(0) = 0$. Proceed by elementary method to show that this solution is $u(t) = \dfrac{e^{\alpha t} - 1}{\alpha + \beta + \beta e^{\alpha}}$ **(06)**

OR

Q.4 (I) The wronskian W(t) of any collection $y_1(t), \ldots, y_n(t)$ of solutions of $y^{(n)}(t) + a_1(t)\, y^{(n-1)}(t) + \ldots + a_n(t)y(t) = 0$ is given by **(07)**

$$W(t) = W(t_0) \exp\left[-\int_{t_0}^{t} a_1(s)\,ds\right]; \text{ for } t, t_0 \in [r_1, r_2)$$

(II) A fundamental system of solutions of $x'(t) = A(t)x(t)$, $x(t_0) = x_0$ exists. **(07)**

(III) Apply Picard's method to solve initial value problem $u' = u + t$, $u(0) = 1$ and show that the successive approximations tends to the exact solution. **(06)**

Q.5 (I) Prove that the solution x(t) of $x'(t) = A(t)x(t) + B(t)$, satisfying $x(t_0) = x_0$, $for\ t \in (r_1,r_2)$.is given by $x(t) = \Phi(t)x_0 + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)B(s)ds$, $r_1 < t < r_2$, where $\Phi(t)$ be a fundamental matrix of $x'(t) = A(t)x(t)$ with $\Phi(t_0) = I$. **(07)**

(II) Let y(t) be any solution of $y^{(n)}(t) + a_1(t)\, y^{(n-1)}(t) + ... + a_n(t)y(t) = 0$ and collection $y_1(t),..., y_n(t)$ be a fundamental system of this equation, then there exists constants $C_1,C_2,.....,C_n$ such that $y(t) = \sum_{j=1}^{n} C_j y_j(t)$ **(07)**

(III) If $g(t,u)$ is continuous function for all $t \geq 0$, $u \geq 0$ show that a function $\phi(t)$ is a solution of $u'' + \mu^2 u = g(t,u)$, $u(0) = 0$ $u'(0) = 1$ where $\mu > 0$ $for\ t \geq 0$ if and only if $\phi(t)$ is a solution of an integral equation **(06)**

$$u(t) = \frac{\sin\mu(t)}{\mu} + \int_0^t \frac{\sin\mu(t-s)}{\mu} g\,(s, u(s))\,ds$$

OR

Q.5 (I) If Φ is a fundamental matrix of $x'(t) = A(t)x(t)$ and let $t_0 \in (r_1, r_2)$ then $W(t) = W(t_0)\exp(\int_{t_0}^{t} Tr\,A(s)ds)$, $t \in (r_1, r_2)$. **(07)**

(II) A necessary and sufficient condition for $y_1(t),..., y_n(t)$ be a fundamental system of solutions of $y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + ... + a_n(t)\,y(t) = 0$ is W(t) $W(t) \neq 0$ for $r_1 < t < r_2$ **(07)**

(III) If K_1 *and* K_2 are positive constants and u is a non-negative continuous function on the interval $\alpha \leq t \leq \beta$ satisfying the inequality, **(06)**

$u(t) \leq K_1 + K_2 \int_{\alpha}^{t} u(s)ds$ show that $u(t) \leq K_1 \exp(K_2(t - \alpha))$.